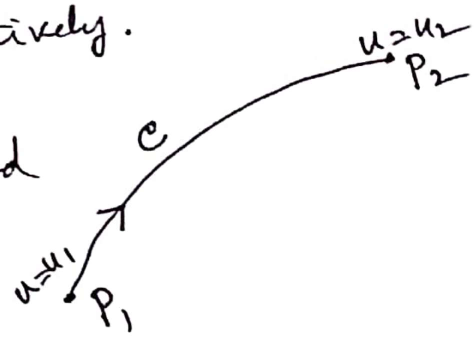
Vector Integration

Ordinary Integrals of vectors: Let $\vec{R}(u) = R_1(u)\hat{i} + R_2(u)\hat{j} + R_3(u)\hat{k}$ be a vector depending on a single scalar variable u, then

$$\int \vec{R}(u)\,du = \hat{i}\int R_1(u)\,du + \hat{j}\int R_2(u)\,du + \hat{k}\int R_3(u)\,du$$

is called an indefinite integral of $\vec{R}(u)$.

Line Integral: Let $\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$, where $\vec{r}(u)$ is a position vector of (x, y, z), define a curve C joining the points P_1 & P_2, where $u = u_1$ & $u = u_2$ respectively.



Let $\vec{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component along C from P_1 & P_2 written as

$$\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_C \vec{A}\cdot d\vec{r} = \int_C A_1dx + A_2dy + A_3dz$$

$$\boxed{\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ d\vec{r} &= dx\hat{i} + dy\hat{j} + dz\hat{k} \end{aligned}}$$

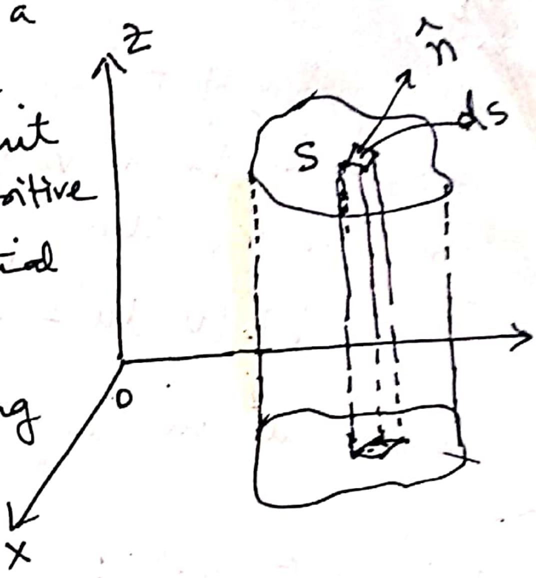
is a Line Integral.

(i) If $\vec{A}$ is the force $\vec{F}$ on a particle along C, then this integral represents the work done by the force.

(ii) If C is the simple closed curve (the curve does not intersect itself anywhere), then the integral around ~~along~~ C is

$$\oint \vec{A}\cdot d\vec{r} = \oint A_1dx + A_2dy + A_3dz$$

(iii) In Aerodynamics and Fluid Mechanics; this integral is called the circulation of $\vec{A}$ about C, where $\vec{A}$ represents velocity of the fluid.



Surface Integral :- Let S be a two sided surface. one side of S is positive side. $\hat{n}$ is the unit normal to any point on the positive side S. Now ds is the differential of surface area. $\underline{ds}$ denotes a vector whose direction is along $\hat{n}$, then $\underline{ds} = \hat{n}\, ds$.

the Integral

$$\iint_S \underline{A} \cdot d\underline{s} = \iint_S \underline{A} \cdot \underline{n}\, ds$$ is an example of surface integral and is called flux of $\vec{A}$ over S.

Volume Integral: Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \underline{A}\, dv \quad \text{and} \quad \iiint_V \phi\, dv$$ are examples of volume Integrals or space space Integrals.

Q-32. P-102. The acceleration $\vec{a}$ of a particle at any time $t \geq 0$ is given by $\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t=0$; find $\vec{v}$ and $\vec{r}$ at any time.

A:- $\vec{a} = \dfrac{d\vec{v}}{dt} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$

Integrating; $\int \dfrac{d\vec{v}}{dt}dt = \hat{i}\int e^{-t}dt - \hat{j}\int 6(t+1)dt + \hat{k}\int 3\sin t\,dt$

$\vec{v} = -\hat{i}e^{-t} - \hat{j}\left(6\dfrac{t^2}{2} + 6t\right) - \hat{k}\,3\cos t + \vec{c}_1$ — (1)

where $\vec{c}_1$ is constant.

when $t=0$; $\vec{v} = 0$;

$\therefore \vec{0} = -\hat{i} - \hat{j}(0) - 3\hat{k} + \vec{c}_1$

$\therefore \vec{c}_1 = \hat{i} + 3\hat{k}$.

From (1); $\vec{v} = -e^{-t}\hat{i} - (3t^2 + 6t)\hat{j} - 3\cos t\,\hat{k} + \hat{i} + 3\hat{k}$

$\vec{v} = (1-e^{-t})\hat{i} - 3t(t+2)\hat{j} - 3(\cos t - 1)\hat{k}$ Ans;

Again $\vec{v} = \dfrac{d\vec{r}}{dt} = (1-e^{-t})\hat{i} - 3t(t+2)\hat{j} - 3(\cos t - 1)\hat{k}$

Integrating; $\vec{r} = \hat{i}\int(1-e^{-t})dt - \hat{j}\int(3t^2+6t)dt - 3\hat{k}\int(\cos t - 1)dt$

$\vec{r} = (t + e^{-t})\hat{i} - \left(\cancel{3}\dfrac{t^3}{\cancel{3}} + \cancel{6}\dfrac{t^2}{\cancel{2}}\right)\hat{j} - 3\hat{k}(-\sin t - t) + \vec{c}_2$ — (2)

where $\vec{c}_2$ is constant.

when $t=0$; $\vec{r} = 0$;

$\vec{0} = (0+1)\hat{i} - (0)\hat{j} - 3\hat{k}(0) + \vec{c}_2$

$\therefore \vec{c}_2 = -\hat{i}$

From (2); $\vec{r} = (t+e^{-t})\hat{i} - (t^3 + 3t^2)\hat{j} + 3\hat{k}(\sin t + t) - \hat{i}$

$\vec{r} = (t - 1 + e^{-t})\hat{i} - (t^3 + 3t^2)\hat{j} + 3(\sin t + t)\hat{k}$ Ans

Q-37. P-102. If $\vec{A} = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$; Evaluate $\int_c \vec{A}\cdot\vec{dr}$ along the Path c;

(a) $x = 2t^2$; $y = t$; $z = t^3$ from $t=0$ to $t=1$.

A:- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

$$\therefore \int_c \vec{A}\cdot\vec{dr} = \int_c \left[(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}\right]\cdot\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$$

$$= \int_c (2y+3)\,dx + xz\,dy + (yz - x)\,dz.$$

Now $x = 2t^2$; $y = t$; $z = t^3$; Then $dx = 4t\,dt$; $dy = dt$ & $dz = 3t^2\,dt$

$$\therefore \int_c \vec{A}\cdot\vec{dr} = \int_{t=0}^{1} (2t+3)(4t\,dt) + 2t^2\cdot t^3\,dt + (t^4 - 2t^2)(3t^2\,dt).$$

$$= \int_{t=0}^{1} (8t^2 + 12t)\,dt + 2t^5\,dt + (3t^6 - 6t^4)\,dt$$

$$= \left[\frac{8t^3}{3} + \cancel{12}^{6}\frac{t^2}{\cancel{2}} + \cancel{2}\cdot\frac{t^6}{\cancel{6}_3} + 3\cdot\frac{t^7}{7} - 6\cdot\frac{t^5}{5}\right]_0^1$$

$$= \left(\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right) - 0$$

$$= \frac{288}{35} \text{ Ans.}$$

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